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ABSTRACT

The expression for the average energy of the two interacting particles undergoing a thermonuclear reaction is developed and results are presented for selected reactions between light isotopes. It is shown that the assumption that the average energy is equal to 3kT is only valid for a 1/v relative cross section dependence.

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I. INTRODUCTION

These are two basic properties of a thermonuclear reaction which are independent of the number, types, or kinematics of the secondary particles. These are the reaction rate and the average energy of the two interacting particles. The reaction rate determines the source strength of suprathermal secondary particles and the depletion of the plasma consituents. Tables of reaction rates for selected reactions of interest are available, e.g., Ref. 1. The average energy of the two interacting particles is required for the overall energy balance. As no such data have been available, this topic will be developed here.

In the following section, the necessary relation for the average energy of the two particles underoing a thermonuclear reaction will be developed, and its physical significance defined. It is often assumed that this quantity is equal to twice the average energy of the Maxwellian distribution, i.e., 3kT, where kT is the ion temperature. It will be shown that this assumption is only valid when the relative reaction rate is constant, i.e., when the relative cross section is inversely proportional to the relative speed. Because of the Coulomb barrier, this energy dependence is never encountered in charged particle induced reactions. Results will be presented for several thermonuclear reactions. Finally, conclusions based on this work will be given.

II. DEVELOPMENT

The expression for average energy conservation in a thermonuclear reaction is given by

$$<(E_1 + E_2)\sigma v\,(kT)> + Q<\sigma v\,(kT)> = \sum_n < E_n\,\sigma v\,(kt)> . \tag{1}$$

The first term of the left hand side of Eq. (1) is the average kinetic energy removal rate of the two interacting particles and the second term is the energy production rate of the reaction. The right hand side is the average kinetic energy appearance rate for all reaction products.

It is the first term on the right hand side of Eq. (1) which is of current interest. Before proceeding, the derivation of the thermonuclear reaction rate will be recalled briefly. Consider particle 1, energy E_1, undergoing a collision with particle 2, which is in a Maxwellian distribution at temperature kT. The inflight (Doppler broadened) reaction rate in this case is given by[2]

$$<\sigma v_1(v_1,kT)> = \left(\frac{M_2}{2\pi kT}\right)^{1/2} \frac{1}{v_1} \int_0^\infty dv_r\, \sigma(v_r)\, v_r^2$$

$$\cdot \left\{ e^{-\frac{M_2}{2kT}(v_1-v_r)^2} - e^{-\frac{M_2}{2kT}(v_1+v_r)^2} \right\} , \tag{2}$$

where M's refer to particle masses and v_r is the relative speed, i.e., $v_r = |\vec{v}_1 - \vec{v}_2|$. If particle 1 is also in a Maxwellian distribution, $N(v_1)$, at temperature kT,

$$N(v_1) = 4\pi \left(\frac{M_1}{2\pi kT}\right)^{3/2} v_1^{\,2}\, e^{-\frac{M_1 v_1^2}{2kT}} , \tag{3}$$

the thermonuclear reaction rate is given by

$$<\sigma v\,(kT)> = \int_0^\infty dv_1\; N(v_1)\; <\sigma v_1(v_1 , kT)> . \tag{4}$$

Using Eqs. (2) and (3) in Eq. (4), inverting the order of integration, and carrying out the integration over v_1 leads to the usual result[1]

$$\langle\sigma v(kT)\rangle = \frac{2^{3/2}}{\pi^{1/2}} \frac{1}{\mu^{1/2}} \frac{1}{(kT)^{3/2}} \int_0^\infty dE_r E_r \sigma(E_r) e^{-\frac{E_r}{kT}} , \tag{5}$$

where μ is the reduced mass; $\mu = M_1 M_2/(M_1+M_2)$. The relative energy, E_r, is given by $E_r = M_2 E_1'/(M_1+M_2)$, where E_1' and $\sigma(E_1')$ refer to the energy and corresponding cross section when the target is at rest.

In determining the average energy of the two interacting particles, the same procedure as just described for the reaction rate will be followed. Analogous to Eq. (4), the average energy removal rate of particle 1 undergoing thermonuclear reactions is given by

$$\langle E_1 \sigma v(kT)\rangle = \int_0^\infty dv_1 N(v_1) \; E_1 \langle\sigma v_1(v_1, kT)\rangle . \tag{6}$$

Equations (2) and (3) are now substituted into Eq. (6) and the order of integration is inverted, which leads to

$$\langle E_1 \sigma v(kT)\rangle = \frac{M_1^{5/2} M_2^{1/2}}{2\pi (kT)^2} \int_0^\infty dv_r \sigma(v_r) \; v_r^2 \; I(v_r , kT) , \tag{7}$$

where

$$I(v_r ,kT) = \int_0^\infty dv_1 v_1^3 \; e^{-\frac{M_1 v_1^2}{2kT}} \left\{ e^{-\frac{M_2}{2kT}(v_1 - v_r)^2} - e^{-\frac{M_2}{2kT}(v_1 + v_r)^2} \right\} . \tag{8}$$

If Eq. (8) is written explicitly as the difference of two integrals, the variable of integration is changed in the second integral, and the results combined, there will result one integral over $-\infty \le v_1 \le \infty$ which is directly integrable.[3] One obtains

$$I(v_r,kT) = (2\pi)^{1/2}(kT)^{3/2} \frac{M_2}{(M_1 + M_2)^{5/2}} v_r \, e^{-\frac{\mu v_r^2}{2kT}} \left(\frac{M_2}{M_1}\frac{\mu}{kT} v_r^2 + 3\right), \tag{9}$$

where μ is the reduced mass. Substituting Eq. (9) into Eq. (7) and reducing yields

$$< E_1 \sigma v(kt)> = \frac{3kT}{2}\left(\frac{M_1}{M_1 + M_2}\right) <\sigma v(kt)>$$

$$+ \left(\frac{M_2}{M_1 + M_2}\right) \frac{2^{2/3}}{\pi^{1/2}} \frac{1}{\mu^{1/2}} \frac{1}{(kT)^{3/2}} \int_0^\infty dE_r \sigma(E_r) E_r^2 \, e^{-\frac{E_r}{kT}}. \tag{10}$$

This analysis is repeated for particle 2 and the result is added to Eq. (10) to obtain

$$<(E_1 + E_2 \sigma v(kT))> = < E_1 \sigma v(kT)> + < E_2 \sigma v(kT)>$$

$$= \frac{3kT}{2} <\sigma v(kT)> + \frac{2^{3/2}}{\pi^{1/2}\mu^{1/2}(kT)^{3/2}} \int_0^\infty dE_r \sigma(E_r) E_r^2 \, e^{-\frac{E_r}{kT}}. \tag{11}$$

The average energy of the two interacting particles undergoing a thermonuclear reaction is then

$$<(E_1 + E_2(kT))> = \frac{<(E_1 + E_2 \sigma v(kT))>}{<\sigma v (kT)>} = \frac{3kT}{2} + \bar{E}_o , \tag{12}$$

where

$$\bar{E}_o = \frac{1}{<\sigma v(kT)>} \frac{2^{3/2}}{\pi^{1/2}\mu^{1/2}(kT)^{3/2}} \int_0^\infty dE_r \sigma(E_r) \, E_r^2 \, e^{-\frac{E_r}{kT}}. \tag{13}$$

We note that Eqs. (12) and (13) agree with the sum of the average final particle energies minus the reaction Q value for the special case of two-body reactions assuming an isotropic center of mass angular distribution [see Eq. (55) of Ref. 4].

The physical significance of the energy, $\bar{E}_o$, is illustrated if the relative cross section is given in the Gamow form[6]

$$\sigma(E_r) = \frac{S(E_r)}{E_r} e^{-b/E_r^{1/2}} , \tag{15}$$

where $S(E_r)$ is the astrophysical S factor, the theoretical factor for b is given by

$$b = 2\pi\alpha \ Z_1 Z_2 \left(\frac{\mu c^2}{2}\right)^{1/2} , \tag{15}$$

and α is the fine structure constant. In units of MeV, $b = 0.990 \ [A_1 A_2/(A_1+A_2)]^{1/2} Z_1 Z_2$ $\mathrm{MeV}^{1/2}$. For nonresonant reactions, e.g., d+d, S(E) can be taken as constant. Under these conditions, the reaction rate integrand, Eq. (5), has a maximum - - the so called Gamow peak. The relative energy region near this peak makes most of the contribution to the reaction rate. The reaction rate integral can be estimated by the method of steepest descent,[7] and is characterized by the energy E_o, where

$$E_o = \left(\frac{kTb}{2}\right)^{2/3} . \tag{16}$$

The relative energy, E_o, is that at the Gamow peak. Similarly, under the added constraint that $E_o >> kT/2$, the integral in the numerator of Eq. (13) may be estimated by the method of steepest descent[4] with the result that $\bar{E}_o = E_o$ as given by Eq. (16). The energy, $\bar{E}_o$, can thus be considered to be the average relative energy which contributes most of the reaction rate.

One would then expect that the other term in Eq. (12), 3kT/2, is the average energy of the center of mass in the laboratory (L) system. To investigate this expectation, assume a two body interaction with an isotropic center of mass (CM) angular distribution yielding reaction products 3 and 4. The quantity to be calculated is E_3, the average energy of reaction product 3. Conservation of average energy and momentum in the CM system yields for particle 3

$$E_3^* = \frac{M_4}{M_1 + M_2} (\bar{E}_o + Q) \tag{17}$$

where $M_1 + M_2 = M_3 + M_4$ and E_3^* is the average energy of particle 3 in the CM system. In terms of velocities, $\vec{V}_3^*$ is now added to $\vec{v}_{cm}$ to obtain $\vec{v}_3$. the velocity of particle 3 in the L system. In this case, $\vec{v}_3^* \cdot \vec{v}_{cm} = 0$, due to the isotropic CM angular distribution, which leads directly to

$$E_3 = \frac{M_3}{M_1 + M_2} \frac{3kT}{2} + \frac{M_4}{M_1 + M_2} (\bar{E}_o + Q) , \tag{18}$$

which is the same as the result obtained from integration of the thermonuclear spectra of particle 3 [Eq. (55) of Ref. 4]. Thus our expectation as to the two terms in Eq. (12) being the average energy of the center of mass in the L system and the average relative energy of the two interacting particles is valid.

If the relative reaction rate is now assumed constant, i.e., $\sigma(E_r)2^{1/2}E_r^{1/2}/\mu_r$ = constant, then

$$\sigma(E_r)\, E_r^{1/2} = \frac{\mu^{1/2}}{2^{1/2}} \sigma(v_r)\ v_r , \tag{19}$$

which can be factored out of the integrals involved in Eq. (13). This leads to

$$< (E_1 + E_2(kT)) > = 3kT. \tag{20}$$

Thus, the assumption that the average energy of the two particles undergoing a thermonuclear reaction is equal to twice the average energy of the thermal Maxwellian distribution is only valid for a constant relative reaction rate. Since the relative cross section is extremely depressed by the Coulomb barrier at low energies, this assumption is never valid for interaction between charged particles.

III. RESULTS

The expression for the average energy of the two interacting particles, Eq. (12), has been programmed assuming a piecewise linear relative cross section - see Appendix A. Calculations were then performed for selected reactions involving isotopes of hydrogen, helium, lithium, and boron. All nuclear data came from the Lawrence Livermore National Laboratory (LLNL) ECPL evaluated charged-particle library.[5] Calculated values (occurring at low temperatures) for which the reaction rate, Eq. (5), is less than 10^{-40} cm^3/sec have been suppressed.

The average energy of the two interacting particles for isotopies of hydrogen and helium is shown in Fig. 1. Except at low temperatures, the behavior is nearly linear. This is in agreement with the results of Eq. (12) and the discussion through Eq. (16).

Referring again to Fig. 1, the average energy of the two interacting particles for both d+d reactions and for the p+t reaction is greater than 3kT. The latter reaction is endoergic with a relative energy threshold of 0.764 MeV. In these cases, the average kinetic energy being removed due to the reaction is greater than the average kinetic energy of two particles in the thermal Maxwellian distribution. Thus the high energy thermal population is being depleted. The energy into the reaction products, $\langle (E_1+E_2(kT)) \rangle$, is thereby enhanced. The converse is true for the d+t and d+^{3}He reactions at high temperatures for which the average energy of the two interacting particles is less than 3kT.

Results involving isotopies of lithium and boron are shown in Figs. 2-5. In each of these reactions, the average energy of the interacting particles is greater than 3kT.

IV. CONCLUSIONS

The average energy ot the two interacting particles undergoing a thermonuclear reaction is given here and results are presented for selected thermonuclear reactions among the isotopes of hydrogen, helium, lithium, and boron. It is shown that the average energy is equal to 3kT when the relative reaction rate is a constant. This circumstance never occurs for charged particle interactions because of the Coulomb barrier. The calculations show that the average energy is almost always greater than 3kT, indicating both a depletion of the high energy tail of the thermal distribution and an energy enhancement for the thermonuclear reaction products. For energy conservation, any model describing the kinematics of the reaction products for thermonuclear reactions should be consistent with the average total available energy, i.e., $< (E_1 + E_2(kT)) > + Q$.

APPENDIX A

If the relative cross section is piecewise linear, i.e.,

$$\sigma(E_r) = a_i + b_i E_r ; \quad E_{r_i} \leq E_r \leq E_{r_{i+1}} \quad , \tag{A.1}$$

then the two quantities of Eq. (11) can be performed analytically. Referring to these two terms and using Eq. (A.1) leads to

$$\frac{3kT}{2} <\sigma v(kT)> = \frac{3kT}{2} C \sum_i \int_{E_{r_i}}^{E_{r_{i+1}}} dE_r (a_i E_r + b_i E_r^2) e^{-E_r/kT} ,$$

and

$$C \int_0^\infty dE_r \sigma(E_r) E_r^2 e^{-E_r/kT}$$

$$= C \sum_i \int_{E_{r_i}}^{E_{r_{i+1}}} dE_r (a_i E_r^2 + b_i E_r^3) e^{-E_r/kT} . \tag{A.2}$$

One is therefore interested in integrals of the type

$$K^n(\gamma) = \int_0^\gamma x^n e^{-x/kT} dx . \tag{A.4}$$

Differentiating Eq. (A.4) by parts leads to the recurrance relation

$$K^n(\gamma) = kT [nK^{n-1} (\gamma) - \gamma^n e^{-\gamma/kT}] , \tag{A.5}$$

with

$$K^o(\gamma) = kT [1 - e^{-\gamma/kT}] . \tag{A.6}$$

Equations (A.5) and (A.6) yield analytic solutions for all necessary values of $K^n(\gamma)$.

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FIGURE CAPTIONS

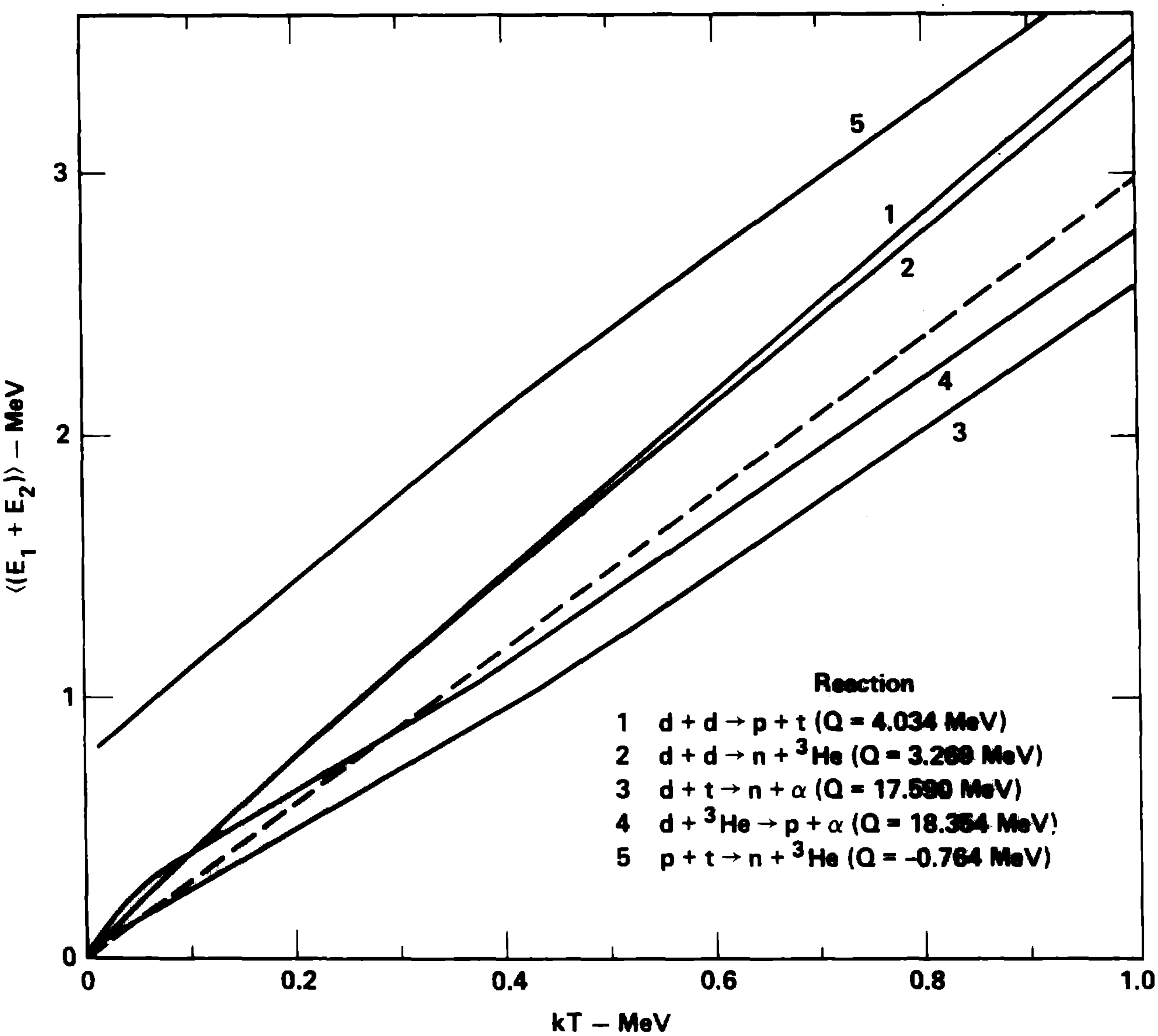
FIG. 1. Average energy of the interacting particles for selected thermonuclear reactions involving hydrogen and helium isotopes. The dashed line corresponds to an average energy equal to 3kT.

FIG. 2. Average energy of the interacting particles for selected thermonuclear reactions involving ^{6}Li and isotopes of hydrogen and helium. The dashed line corresponds to an average energy equal to 3kT.

FIG. 3. Average energy of the interacting particles for selected thermonuclear reactions involving ^{7}Li and isotopes of hydrogen and helium. The dashed line corresponds to an average energy equal to 3kT.

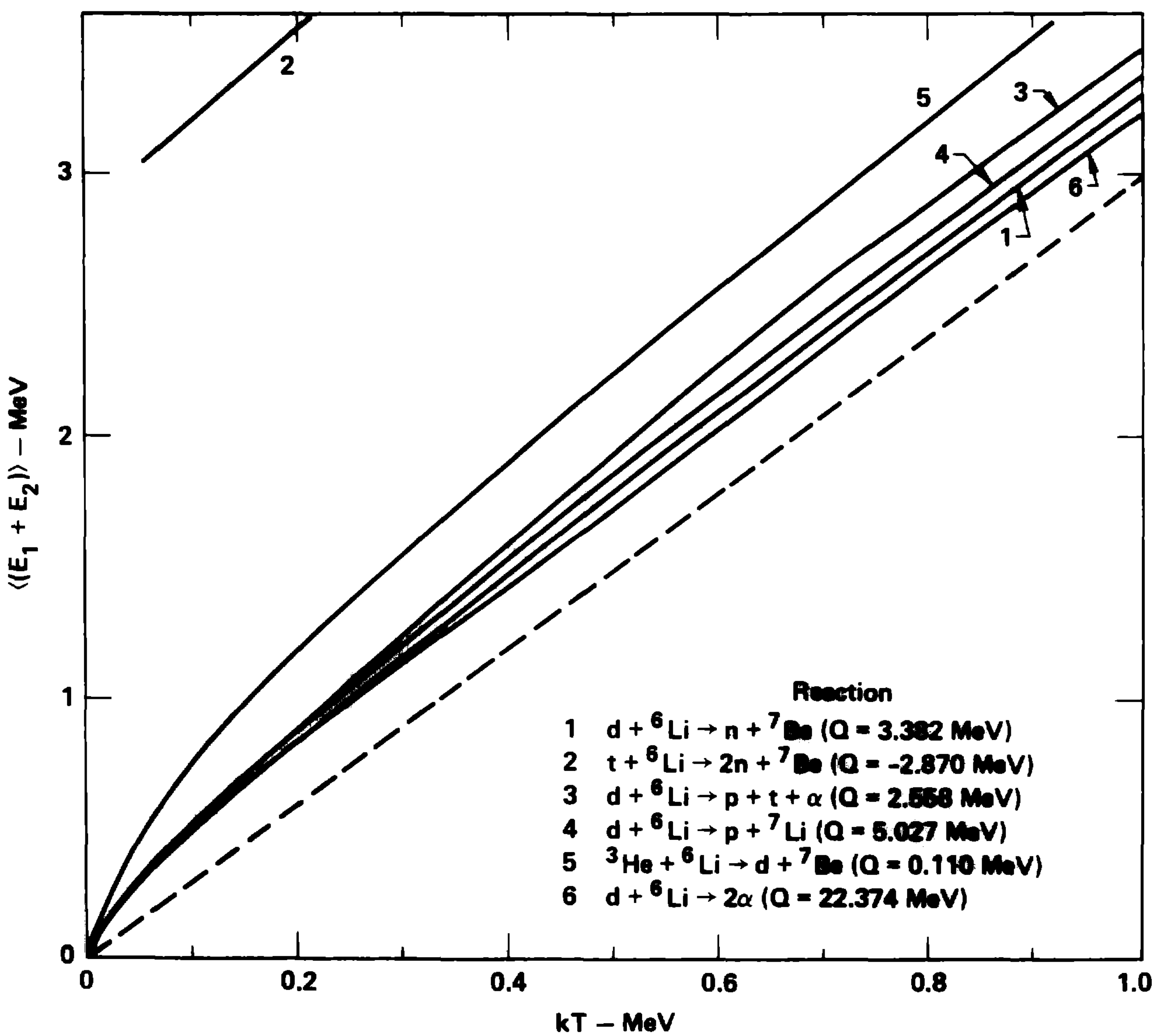
FIG. 4. Average energy of the interacting particles for selected thermonuclear reactions involving ^{10}B and isotopes of hydrogen. The dashed line corresponds to an average energy equal to 3kT.

FIG. 5. Average energy of the interacting particles for selected thermonuclear reactions involving ^{11}B and isotopes of hydrogen. The dashed line corresponds to an average energy equal to 3kT.



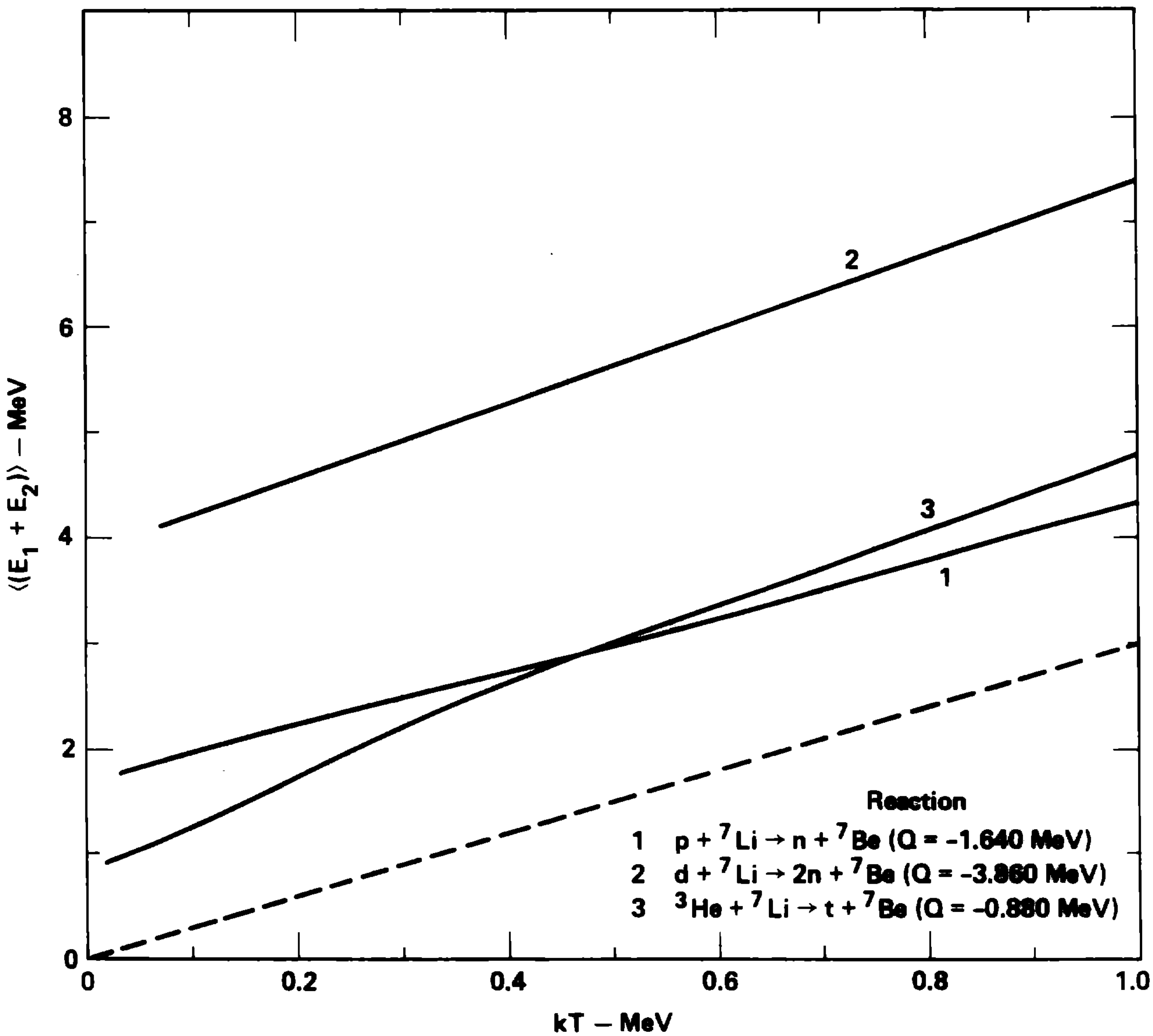
5
1
2
4
3
Reaction
1 d + d → p + t (Q = 4.034 MeV)
2 d + d → n + ³He (Q = 3.269 MeV)
3 d + t → n + α (Q = 17.590 MeV)
4 d + ³He → p + α (Q = 18.354 MeV)
5 p + t → n + ³He (Q = -0.764 MeV)
0
1
2
3
0
0.2
0.4
0.6
0.8
1.0
kT – MeV
⟨(E₁ + E₂)⟩ – MeV

FIG. 1



2
5
3
4
1
6
3
2
1
0
$\langle (E_1 + E_2) \rangle$ — MeV
0
0.2
0.4
0.6
0.8
1.0
kT — MeV
Reaction
1 $d + {}^6Li \rightarrow n + {}^7Be$ (Q = 3.382 MeV)
2 $t + {}^6Li \rightarrow 2n + {}^7Be$ (Q = -2.870 MeV)
3 $d + {}^6Li \rightarrow p + t + \alpha$ (Q = 2.558 MeV)
4 $d + {}^6Li \rightarrow p + {}^7Li$ (Q = 5.027 MeV)
5 ${}^3He + {}^6Li \rightarrow d + {}^7Be$ (Q = 0.110 MeV)
6 $d + {}^6Li \rightarrow 2\alpha$ (Q = 22.374 MeV)

FIG. 2



8
6
4
2
0
0
0.2
0.4
0.6
0.8
1.0
kT — MeV
⟨⟨E1 + E2⟩⟩ — MeV
2
3
1
Reaction
1 p + 7Li → n + 7Be (Q = -1.640 MeV)
2 d + 7Li → 2n + 7Be (Q = -3.860 MeV)
3 3He + 7Li → t + 7Be (Q = -0.880 MeV)

FIG. 3

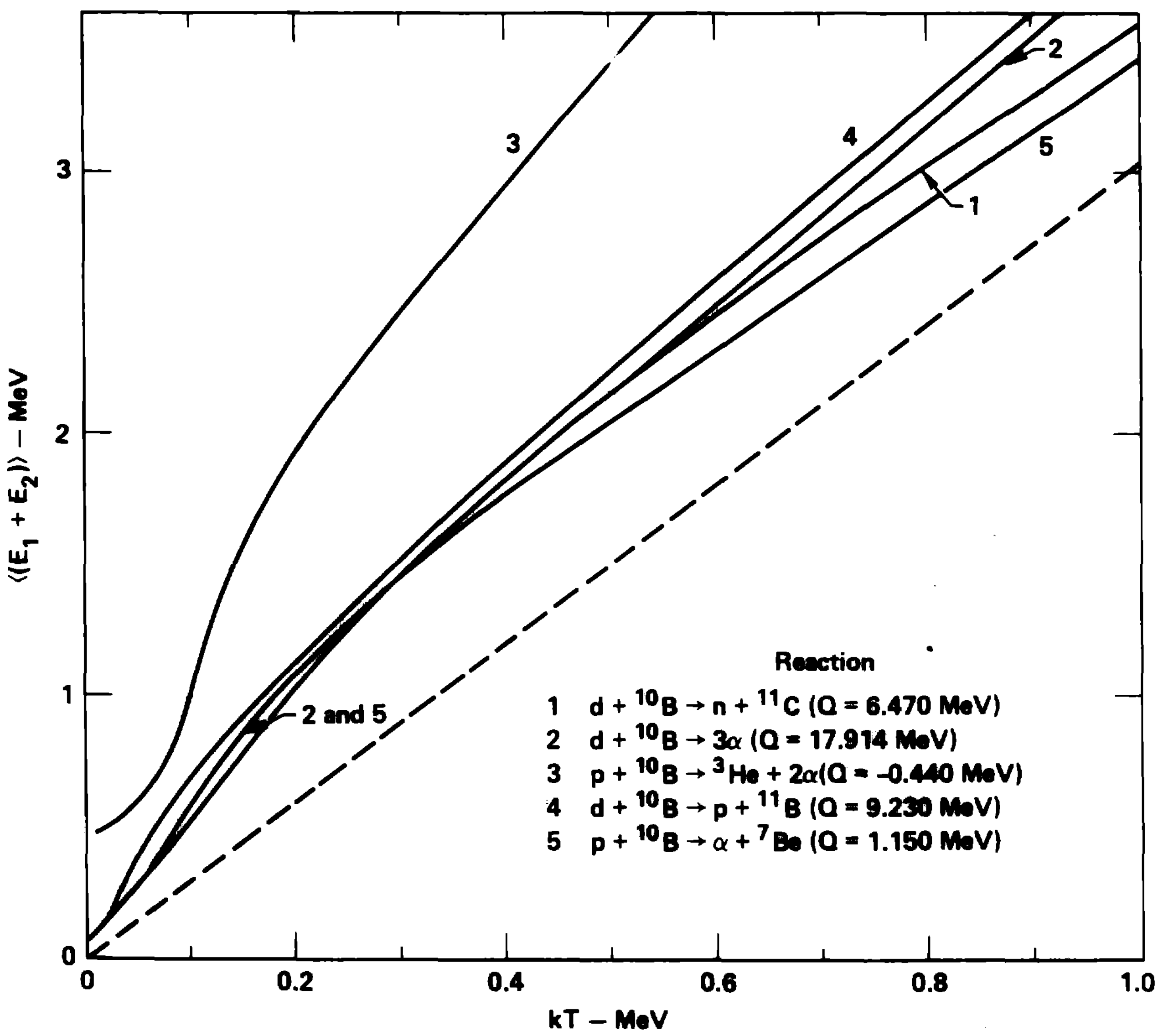

⟨(E1 + E2)⟩ − MeV
kT − MeV
0
0.2
0.4
0.6
0.8
1.0
1
2
3
1
2
3
4
5
2 and 5
Reaction
1 d + 10B → n + 11C (Q = 6.470 MeV)
2 d + 10B → 3α (Q = 17.914 MeV)
3 p + 10B → 3He + 2α(Q = -0.440 MeV)
4 d + 10B → p + 11B (Q = 9.230 MeV)
5 p + 10B → α + 7Be (Q = 1.150 MeV)

FIG. 4

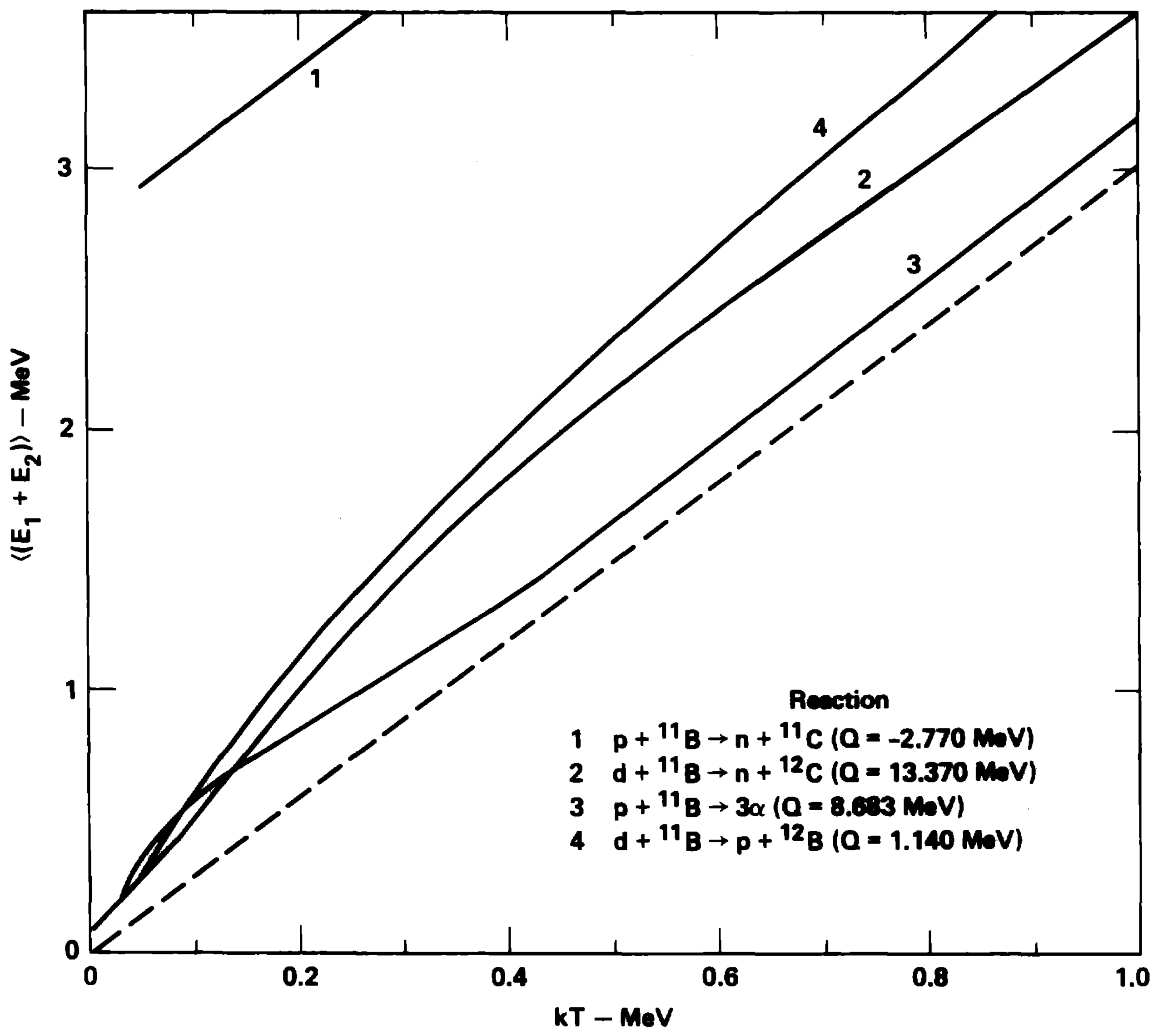

1
4
2
3
3
2
1
0
0
0.2
0.4
0.6
0.8
1.0
kT — MeV
⟨(E1 + E2)⟩ — MeV
Reaction
1 p + 11B → n + 11C (Q = -2.770 MeV)
2 d + 11B → n + 12C (Q = 13.370 MeV)
3 p + 11B → 3α (Q = 8.683 MeV)
4 d + 11B → p + 12B (Q = 1.140 MeV)

FIG. 5